The Inverse Square Law of Gravitation:
An Alternative to Newton's Derivation [1]

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Abstract

The view has long been held by historians of science, that *Sir Isaac Newton's* original derivation of the *inverse square law of gravity*, whilst certainly not lacking brevity, most definitely provides little indication of the original thought processes that led him to the final results. For approximately three hundred years scholars have painstakingly ploughed through the original proofs and have almost unanimously found them difficult[2]; modern scholars are no exception [5, pp54]. In addition, it has been argued, and Newton tended to encourage the idea, that he must have used *The Calculus* to have arrived at his results, and only then worked out his geometric proofs.
This paper presents a modified version of Newton's proof of the inverse square law of gravity, as presented in *Proposition XI, Problem VI* of his *Philosophiae Naturalis Principia Mathematica* [15] (now almost universally known as the *Principia*). The derivation uses the same traditional geometrical approach that Newton used, however, the line of reasoning is considered to be more straight forward than that presented by Newton and, it is believed that it may represent the way he actually arrived at this monumental discovery. It also shows that it is more likely than not that Newton did actually arrive at his results using only *geometrical constructs.*
The general applicability of *Kepler's second law* is first demonstrated as set out in the Principia for a body subject to any *central force.* The consequences of this law when applied specifically to *elliptical, hyperbolic* and *parabolic orbits* are remarkable and, lead directly to proof of the inverse square law of gravity.

Part I - Elliptical Orbits

1 Introduction

The methods of Mathematical Physics are now so sophisticated that problems which taxed the greatest minds centuries ago, can now be solved by under graduates using only a few lines of vector calculus. However, it is instructional to occasionally look at classical problems from the past in order to try and establish the thought processes of the mathematical giants who solved problems that no-one else could, using tools that today we consider basic. It is also worth noting that whilst modern mathematical hieroglyphics provide great economy when describing a particular problem, they also tend to disguise the fact that behind many simple statements lie the accumulated intellectual efforts of previous generations of mathematicians, which have taken place over thousands of years.

[1] This paper is an extended and revised verion of a 1997 Report by the author. [8]

[2] D. Gjertsen reports that "Sir Arthur Eddington and Bertrand Russell both considered the Principia to have been mastered by only a hand full of people, most of whom are either dead or have been driven mad by the [intellectual] effort" [11, pp486].

There are two advantages to be gained by reviewing great works of past masters: first, it may chasten us into realising that, after all, we are not as clever as we would like to believe; and, secondly, some of their genius may rub off and give us new insights so that we may solve modern problems that hitherto have not yielded to analysis. A recently published book written in this spirit is: *Journey Through Genius*, by William Dunham [3], which is a highly recommended source of such problems and their historical background. This paper looks at one such problem, namely, proof of the inverse square law of gravity.

It is my belief that if we loose the ability to stand in awe and wonder when confronted with products of the creative mind, we tend to become self centred and cynical. The investigation of the cause and effects of gravity has a history that very few other problems have, and many people regard it as mankind's greatest intellectual quest. It started in antiquity, even before man could write and pass on his thoughts to the next generation, and it has continued up until the present day.

The study of the attraction of matter and its link with mathematical astronomy and the motions of the heavens, is also a truly international problem that has been addressed by most, if not all, the very great mathematical and philosophical minds of yore. Newton acknowledged his debt to his scientific predecessors in his now famous phrase: *"If I have seen further, it is by standing on ye shoulders of giants,..."* [19, p274]. which he wrote in a letter to Robert Hooke in 1676, when they were still, apparently, on amicable terms.

The astronomical work of antiquity was carried out primarily by Plato, Aristarchus, Aristotle, Appollonius, Hipparchus and Ptolemy. The work of these original thinkers resulted in acceptance of the Geocentric or Ptolemaic system, which was finished off and recorded by Ptolemy in his renowned work the *Almagest* in circa 150 A.D. This great work became the bible for astronomers, and remained so until 1543, when Copernicus(1473-1543) began the final assault on solving the mystery with the introduction his heliocentric view of the Solar System, which was only published in his *De Revolutionibus Oribium Celestium* in the year of his death, though the work had been completed some ten years previously (in this connection, it should be mentioned that Aristarchus had proposed a heliocentric system in circa 270 B.C.). Galileo(1564-1642) followed suit by finding confirmation of Copernicus' theory in his astronomical observations, which were published in his *Sidereus Nuncius* in 1610. He was later imprisoned by The Inquisition for supporting the Copernican Heliocentric System in his *Dialogue on the Two Chief Systems of the World, Ptolemaic and Copernican* which he published in 1632, and forced to recant at his infamous trial in 1633. Even though no serious challenges to the Heliocentric System were mounted after the late seventeenth century, it is indeed remarkable that Galileo has only recently been exonerated by the Catholic Church from being a heretic [6]. Galileo is also credited with having started the long process of developing a scientifically sound basis for the theory of classical mechanics. This detective work was continued by Tycho Brahe(1546-1601) who spent most of his scientific career making meticulous observations of the planets, and in particular Mars. This data later enabled Tycho Brahe's assistant Johannes Kepler(1571-1630) to formulate his three fundamental laws, the first two of which were published in his *Astronomia nova* and the third in *Harmonice mundi* in 1609 and 1619 respectively. It was these three laws which really finalised the physics ground work, see Appendix I.

Most of the mathematics necessary for completing the picture had been around for a very long time. This consisted of fundamental geometry as defined in circa 300 B.C. by Euclid in his *Elements of Geometry,* together with the theory of conic sections as laid down in circa 225 B.C. by Appollonius in his *On Conic Sections.* Further mathematical ground work was laid by Pierre de Fermat, Isaac Barrow and others, with Descartes developing *Cartesian Geometry* which he published in his *Gï¿œomï¿œtrie* in 1637. This was the final tool that would enable Newton

(1642-1727) to complete the picture so magnificently with the formulation of his *Universal Law Of Gravitation* and *System Of The World*, which were first published in *The Principia* in 1687. The work of Newton was then continued by MacLauran, Lagrange, Laplace, Legendre, Gauss, Bessel, Ivory and other worthies too many to mention. Newton's work has stood the test of time and his laws are still used for the vast amount of mechanical calculations performed every day. However, it appears that no theory is perfect, and this was shown to be the case when Einstein demonstrated that Newton's laws had to be modified for speeds approaching that of light. When Einstein published his *Special Theory of Relativity* and his *General theory of Relativity* in 1905 and 1915 respectively, the Newtonian era was brought to a close. Even during the twentieth century many of our best minds have pondered the subject of gravity, e.g. Bondi, Hawking, Sagen, Wheeler, etc. It continues today, though the emphasis now is on answering fundamental cosmological questions involving *General Relativity*.

The concept of an inverse square law for gravity had also been postulated independently by Sir Christopher Wren, Dr Robert Hooke, and Dr Edmond Halley as acknowledged by Newton in the *Principia* [15, p46], but without them offering any mathematical or scientific reasoning. However, the man most central to the first complete solution of this problem is the author of the *Principia*, Sir Isaac Newton, whose biography has been so admirably told by R. S. Westfall in his book: *Never At Rest* [19]. We, the followers of Newton, are extremely fortunate as many of the great works leading up to and including the *Principia* are available as inexpensive reprints; so we are able to own and to appreciate these masterpieces at first hand. It is appropriate to note that Newton had apparently deduced previously that gravity obeyed an inverse square law, in 1666, i.e. almost twenty years prior to publication of the *Principia*[3].

The *Principia* stands as one of the greatest scientific works ever published because it contains original mathematical and scientific accomplishments of unprecedented greatness. At a stroke it pushed back the frontiers of knowledge so far, and on such a broad front that the scientific community was sent reeling for many years. All over the world scholars hailed it as a masterpiece the like of which had not been seen before (and probably never will be again).

Newton was born in Woolsthorpe Manor, near Grantham, England in 1642, the same year that Galileo died. He reached the grand age of 84 years, which in those days was its self a major accomplishment. During most of his active scientific life he was The Lucasian Professor of Mathematics at Cambridge University. He became president of The Royal Society in 1703 and remained in the post until his death in 1727. During the later part of his life he retired from Cambridge and became Master of the Royal Mint. He was also knighted in 1705 by Queen Anne. In all the posts that he occupied, he excelled and made substantial contributions. In addition to writing the *Principia*, he is well known for having been the first to have invented the calculus (which was also invented independently by Leibnitz) and to have developed the first coherent theory of Optics. However, even this great man was not above internecine disputes with fellow scientists, and in particular with Leibnitz in respect of priority claims over the invention of the calculus. Each proponent amassed an army of followers, but neither side emerged with their reputations untarnished, and an excellent review of this episode can be found in A. R. Hall's book: *Philosophers At War* [10]. Still, perhaps this is an undercurrent that will always pervade the creative process; because, when reputations are at stake, fury reigns [4, p92]. For instance, consider the recent unedifying priority dispute between the French and Americans over which group first discovered the AIDS virus [12, 1].

[3] This he did by considering the centrifugal force associated with a circular orbit (v^2/r) and applying Kepler's Third Law ($r^3/T^2 = k$) [7, p238]. Although it is now known that Newton was first to discover how to calculate centrifugal force, Huygens was first to publish the result, in his *Horologium Oscillatorium* of 1673, and therefore gained priority.

Newton's laws of Motion have been included as Appendix II because they were fundamental to the discovery of the inverse square law of gravity and, consequently, are central to the derivations which follow.

2 Derivation of Kepler's 2nd Law (General)

Kepler formulated his First and Second Laws in 1609 from data derived from astronomical observations carried out by Tycho Brahe and himself. The Second Law can be stated in modern mathematical form as,

$$\frac{dA}{dt} = \text{constant} \tag{1}$$

where,

A = area swept out by radius vector, $[m^2]$
t = time, $[s]$.

Now consider figure (1) below.

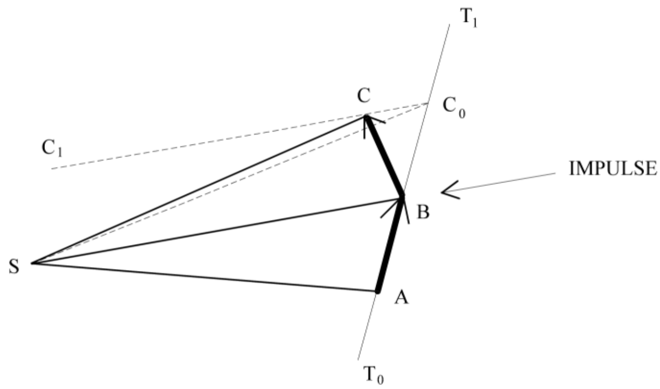

Figure 1: Demonstration Of Kepler's 2nd Law

According to Newton's 1st Law (Appendix I), if a body is moving in a straight line at a constant speed along the path from T_0 to A then, unless it is subjected to a net force, it will continue along this path indefinitely, eventually passing through point T_1 and beyond. Thus, the body will move at a constant speed from A to B in a given time. Without an external influence, it would arrive at point C_0 after a further period of the same time. However, if an impulse is applied at point B then, by the parallelogram rule for forces and velocities, the body will be deflected to point C, and the line CC_0 will be parallel to line SB. It is clear therefore, that the areas of $\triangle SBC$ and $\triangle SBC_0$ are equal - due to the impulse being directed towards the fixed point S, giving rise to triangles with equal bases and heights. It is also clear that the area of $\triangle SAB$ is equal to that of $\triangle SBC_0$. Thus, it follows that this result can be extended by similar argument to a series of impulses, see below. An impulse can be defined as a force applied for an infinitesimal period of time, such that the result is an instantaneous change in velocity and, hence momentum. In figure (2), which represents an approximation to a central gravitational force, if the time period is shortened to an infinitesimal period, the series of straight lines will

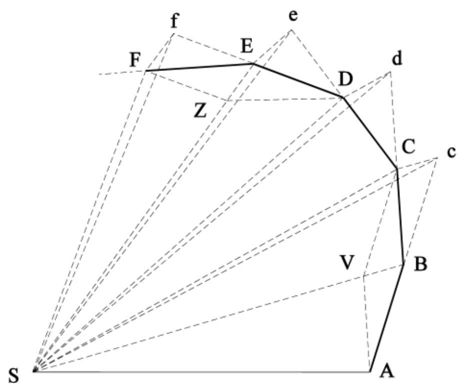

Figure 2: Newton's Original Diagram

become an infinitesimally varying curve, and the impulses will become effectively a smoothly varying force - though this is not a necessary condition for the validity of Kepler's 2nd Law.

The diagram in figure (2), was first published in 1684 in Newton's 9 page treatise *De motu corporum in gyrum* [19, p412], and forms the basis of Proposition I, Theorem I of the *Principia*.

Thus, it follows that if the lengths AB, BC,$\cdots$, EF represent the distances travelled in equal times, then, following a similar line of reasoning to the above, the areas $\triangle SAB$, $\triangle SBC$,$\cdots$, $\triangle SEF$ will all be equal. This demonstrates the validity of Kepler's 2nd Law. Also, as the impulse forces are the only forces acting on the body, and because they always act towards S, the segments AB, BC,$\cdots$, etc. will all lie in the same plane.

Whilst the above is a simplified treatment, and ignores without proper justification certain subtleties in connection with 2nd order infinitesimals involved in the limiting case when P and Q coincide (see figure 4), nevertheless, it does arrive at the correct result - this aspect has been fully analyzed by D. T. Whiteside [20, pp35-7].

These are indeed remarkable results. Furthermore, because no restrictions have been placed upon the type of force, it also follows that Kepler's 2nd Law actually applies regardless of the form of central force, i.e. the force does not necessarily have to conform to an inverse square law. This is perhaps the most remarkable result, which Newton also demonstrated in Proposition I, Theorem I.

3 Kepler's 2nd Law Applied to Elliptical Orbits

The effect of Kepler's 2nd Law is illustrated for elliptical orbits in figure (3), where the shaded areas represent equal areas described by the radius vector in equal times. This of course requires that the orbital speed varies according to position, i.e. the body moves faster when closer to the focus and slower when further away.

We can now analyse the elliptical orbit in more detail, though differently to Newton. From figure (4) below, where θ is taken to be very small, we can state Kepler's 2nd law mathematically as,

$$\begin{aligned} Area\,(\triangle SPQ) &= \frac{1}{2} SP \cdot QT = \frac{dA}{dt} \times \text{time} \\ \therefore Area\,(\triangle SPQ) &\propto \text{time} \end{aligned} \tag{2}$$

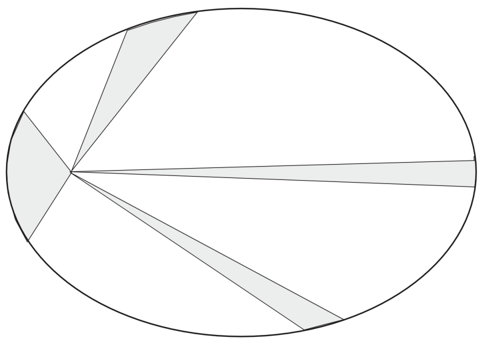

Figure 3: Equal Areas Described In Equal Times

If we now consider a body moving in an elliptical orbit subject only to a central force directed towards one focus S, then in the time it takes for the body to move from point P to point Q in figure (4) below, the line joining the focus S to P will have moved through an angle θ. If the body had not been subject to any force, then due to Newton's first law, it would have moved from point P to point R. However, there is a force acting upon the body which Newton referred to as the *centripetal force*, and it is always directed towards S. Therefore, this force must accelerate the body through the distance QR in the time it takes for SP to move through the angle θ.

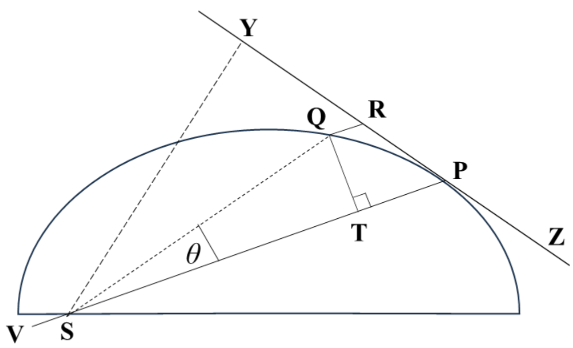

Figure 4: The Path Of A Body In An Elliptical Orbit

Figure (4) is a modified version of the diagram appearing in Proposition VI, Theorem V of the *Principia*.

From *Newton's second law* we see that,

$$QR = \frac{1}{2} f \times t^2 \tag{3}$$

where, f = acceleration towards the focus, $[m/s^2]$
t = time for the body to travel from P to Q, $[s]$

$$\therefore f \propto \frac{QR}{t^2}. \tag{4}$$

Thus, on substituting equation (2) into equation (4) we obtain,

$$f \propto \frac{QR}{SP^2 \cdot QT^2}. \tag{5}$$

It is clear therefore, that if the ratio of QR to QT^2 is constant, then the *inverse square law* follows directly; this is established in section 4.0.

4 Newton's Derivation of The Inverse Square Law

The following derivation (in box) is as set out in Cajori's revised version of Motte's translation of the third edition of Newton's Principia [15, pp56-7]. Figure (5) is a modified version of Figure (4), where the *conjugate diameters* DK and GP have been added, together with additional geometrical lines that are required for the analysis that follows. It has been taken directly from Proposition XI, Problem VI of the *Principia* (except for figure caption) and, for clarity, it should be noted that point x lies on the line SP and point v lies on the line GP; Qv is parallel to RP and IH is parallel to DK. PF and QT are perpendicular to DK and SP respectively.

SECTION III

The motion of bodies in eccentric conic sections.

PROPOSITION XI. PROBLEM VI

If a body resolves in an ellipse; it is required to find the law of the centripetal force tending to the focus of the ellipse.

Let S be the focus of the ellipse. Draw SP cutting the diameter DK of the ellipse in E, and the ordinate Qv in x; and complete the paralellogram $QxPR$. It is evident that EP is equal to the greater semiaxis AC: for drawing HI from the other focus H of the ellipse parallel to to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half-sum of PS, PI,

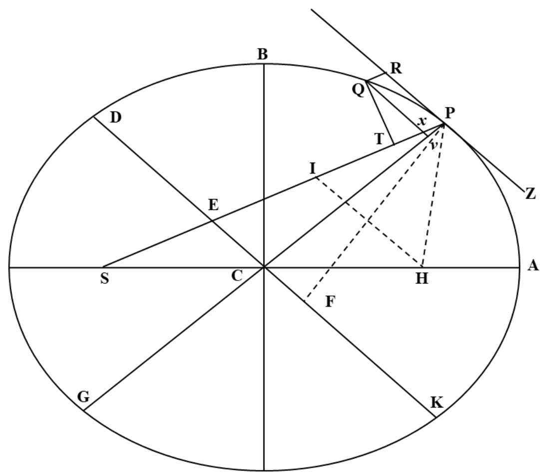

Figure 5: Geometry Of An Elliptical Orbit

that is (because of the parallels HI, PR, and equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis $2AC$. Draw QT perpendicular to SP, and putting L for the principle latus rectum of the ellipse (or for $\dfrac{2\,BC^2}{AC}$), we shall have

$$L \cdot QR : L \cdot Pv = QR : Pv = PE : PC = AC : PC,$$

$$\text{also, } L \cdot Pv : Gv \cdot Pv = L : Gv, \text{ and, } Gv \cdot Pv : Qv^2 = PC^2 : CD^2.$$

By Cor. II, Lem. VII, when the points P and Q coincide, $Qv^2 = Qx^2$, and Qx^2 or $Qv^2 : QT^2 = EP^2 : PF^2 = CA^2 : PF^2$, and (by Lem. XII) $= CD^2 : CB^2$. Multiplying together corresponding terms of the four proportions, and simplifying, we have

$$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2 = 2\,PC : Gv,$$

since $AC \cdot L = 2\,BC^2$. But the points Q and P coinciding, $2\,PC$ and Gv are equal. And therefore the quantities $L \cdot QR$ and QT^2, proportional to these, will be also equal. Let those equals be multiplied by $\dfrac{SP^2}{QR}$, and $L \cdot SP^2$ will become equal to $\dfrac{SP^2 \cdot QT^2}{QR}$. And therefore (by Cor. I and V, Prop. VI) the centripetal force is inversely as $L \cdot SP^2$, that is inversely as the square of the distance SP. Q.E.I.

5 An Alternative Derivation of The Inverse Square Law

We are now in a position to look at an alternative to Newton's original derivation of the inverse square law based upon an elliptical orbit.

The above derivation that Newton included in Proposition XI, Problem VI of the Principia is generally considered to be abstruse[4] and is unlikely to be representative of the process by which he actually arrived at his important discovery. Therefore, the natural approach for any researcher seeking to illuminate this matter would be first to search for material that predates *De motu* and which contains Newton's original derivations, and then to show how he recast this work into the form included in the Principia. Alas, this approach appears to be doomed to failure as, apparently, no such papers have been discovered [20] since Newton's death in 1727 - in this connection, Westfall writes in his definitive biography of Newton [19, p424]: *"No such papers demonstrating propositions of the Principia have ever been found except for a few in which he later set a couple of propositions over into analytical* [i.e. fluxional] *terms".*

Westfall confirms his rejection of the notion that Newton may have used the calculus to arrive at his results by commenting perceptively [19, p424]: *"The problem with the mathematics of the Principia, then, is not to look for prior demonstration in a different form but to see thought patterns of the calculus behind the façade of classical geometry".*

Thus, reluctantly, we have to accept that documents which may shed light on the background to Newton's original thought processes when he first derived the Inverse Square Law are not available for us to peruse. Consequently, in order to gain further insight into the most likely sequence of steps that lead Newton from intuition to knowledge, we have to refer to documents that are available. The primary source documents available to researchers, and which are of particular relevance to this problem are: *De motu corporum in gyrum*[5], *The Principia, 3rd Edition*[6], and the *Letter from Newton to John Locke (March 1689/90)*[7].

From even a superficial analysis of Proposition XI, Problem VI, it is clear that the introduction of L, the *principle latus rectum of the ellipse*, in the way that Newton does at the very start of his derivation, is artificial and must have required a priori knowledge on Newton's part as to the final outcome of the calculation. It would most certainly have been more natural for it to have been introduced at the end of the calculation. I therefore believe that we can safely conclude that the derivations that Newton provided both in the *Principia* and in *De motu*, do not represent the way in which his thought processes evolved at first discovery.

John Locke first read the *Principia* whilst in exile in Holland and only returned to England

[4] As Newton intended [19, p459]

[5] Available in the original Latin together with an English translation and commentary: (i) by D. T. Whiteside [20] with various revisions; (ii) by A. R. Hall and M. B. Hall [9]; and, (iii) by J. W. Herivel [11].

[6] Available: (i) in the original Latin with commentary by A. Koyré and I. B. Cohen [14]; and, (ii) as an English translation by R. Cotes, revised by F. Cajori [15].

[7] Available in the original English with commentary: (i) by H. W. Turnbull [18, pp71-7]; (ii) by A R Hall and M B Hall [9, pp293-301]; and, (iii) by J. W. Herivel [11, pp246-254]. Two versions are available, one in Newton's hand and one, a copy (with some differences), apparently in the hand of Locke's valet and amanuensis Brownover. Herivel offers some persuasive arguments in terms of circumstantial evidence (notably, the total lack of any reference to the Principia, which it would have been natural to cite if it was meant to clarify parts of it) that the original of this document may have been written earlier; either, during the winter of 1679-80 following correspondence with Hooke or, after the first version of the tract *De motu* which then could mean that it represents the original of the first set of propositions carried to Halley by Paget from Cambridge to London in November 1684 in fulfillment of the promise he made to Halley in August of the same year. That this remains an unresolved controversy is apparent from the fact that Westfall [19, pp387-8] supports the view that the letter was written around 1680, whilst Whiteside [20, pp553-4] considers speculation as to the correct date of the text to be "...ultimately not very rewarding".

in February 1689 after having discussed the accuracy of Newton's geometry with Huygens. He must therefore have received the letter from Newton some time later (generally agreed to be March 1690). Thus, if as mentioned above, the letter was based upon a prior proof, it must have been copied from an earlier document already in existence. The proof contained in this letter seems to be a more natural calculation as it does not introduce L artificially at the start of the derivation. I believe that this fact, regardless of whether or not the letter truly predates *De motu*, is significant as it shows that Newton actually derived a proof of Proposition XI, Problem VI of the *Principia* that did not introduce L at the start.

It was perceived by Newton, and others at that time, that the force of gravity on a body should vary according to the inverse of the square of the distance from the centre of force to that body. Hence, Newton would have noticed immediately that the only requirement for the force of gravity to vary inversely as the square of distance for a given orbit is, that the quantity QR/QT^2 must be constant - refer to section 3.0. This could not have been missed by Newton when he first set out to achieve complete philosophical comprehension of the problem. Following proof of Kepler's 2nd Law, Newton's realisation that QR/QT^2 had to be constant for any orbit, must therefore be the prime candidate for the first clue that he solved in his search for proof of the Inverse Square Law. By recognition of this fact, the problem is at once transformed into a more simple task. However, its solution still required the application of sophisticated geometric constructs that, at that time, were only known to Newton. If we accept this premise, then I believe that there is one fundamental calculation sequence that, logically, would have lead Newton to his solution.

Before moving on to reconstruct this sequence, it is appropriate to note that a prior crucial step which enabled Newton to analyze orbits, was his construction of figure (4) and the famous figure (5). These far sighted geometric constructions, which are central to the proof presented in the *Principia*, enabled him to visualise the process in clear mathematical terms and, indeed, in terms that were quite revolutionary for that time, i.e. using the concepts of curvilinear motion[8] and a limit, if not direct application of the calculus.

Westfall also provides an apt comment upon Newton's derivation of the results of Proposition XI, Problem VI [19, p425]: *"Newton's strategy in finding the centripetal force required for motion in a curve, such as an ellipse, was to start with the known properties of the curve and to work towards an expression of the distance of any point P from the centre of force in terms of* $(SP^2 \cdot QT^2)/QR$*. The strategy required that he implicitly redefine the characteristics of curves in terms of ultimate ratios about the points Q and P as they approach each other"*.

Newton begins his three primary published derivations with some fundamental concepts relating to conic sections, prior to providing proofs of the Inverse Square Law. It is most likely that all these concepts were actually known to Newton prior to the derivation, and not learned or discovered at the same time. However, he does make the point of stating explicitly that $EP = CA$ (see *step 1* below), and Newton may have been the first person to state this in writing[9].

Once Newton had proved that QR/QT^2 was indeed constant and equal to $1/L$, he could then set about constructing an elegant, if abstruse, alternative proof for subsequent publication.

In order to demonstrate the required result, it is necessary to transform QR/QT^2 into a form

[8] I. B. Cohen attributes Newton's approach to the analysis of curvilinear motion to ideas provided by Robert Hooke during their correspondence of 1679/80 [2, p218]. Cohen cites a question from Hooke asking Newton to comment on what Hooke called a "hypothesis ... of mine... of compounding the celestiall motions of the planetts[out] of a direct motion by the tangent & an attractive motion towards the centrall body."

[9] This view has been advanced by D T Whiteside [21, p47] as no prior proof of this statement has been found in any previous work on conics.

that consists of known constants. This can be achieved by using the fundamental relationships that Newton defines at the beginning of Proposition XI, Problem VI, and independently transforming QR and QT into alternative forms so that they can be combined in such a way as to lead to the desired proof. The derivation given below (using modern mathematical notation) forms a natural calculation sequence, and, I believe, is more likely to be the route by which Newton first obtained his solution rather than either, that presented in the *Principia* or, one he would have found by using the calculus.

STEP 1 (derive expressions for QR and QT)

It is seen from figure (5) that $\triangle xPv$ and $\triangle EPC$ are similar with their shapes being independent of the position of Q,

$$\therefore Px = QR = Pv \cdot \frac{EP}{CP}. \tag{6}$$

Also, from the fundamental geometric properties of an ellipse, angle $\widehat{RPS}$ is equal to angle $\widehat{ZPH}$. It therefore follows that, for IH parallel to DK and $SP + PH = 2AC$, $IP = PH$ is equal to PH and EP is identically equal to CA [15, p56],

$$\therefore QR = Pv \cdot \frac{CA}{CP}. \tag{7}$$

From the parallelogram $QRPx$ we note that angle $\widehat{QRP}$ is equal to angle $\widehat{QxP}$, from which it follows that angle $\widehat{EPF}$ is equal to angle $\widehat{xQT}$. Thus, $\triangle QTx$ and $\triangle PFE$ are also similar with their shapes independent of the position of Q. It therefore follows that,

$$QT = Qx \cdot \frac{FP}{EP} = Qx \cdot \frac{FP}{CA}. \tag{8}$$

STEP 2 (combine above to obtain an expression for QR/QT^2)

From the above it is seen that,

$$\frac{QR}{QT^2} = \frac{Pv \cdot CA^3}{Qx^2 \cdot CP \cdot FP^2} \tag{9}$$

STEP 3 (Obtain simplifying expression based upon conjugate diameters)

From Appendix III it is seen that

$$\frac{Gv \cdot Pv}{Qv^2} = \frac{CP^2}{CD^2}. \tag{10}$$

Note: CP and CD are *conjugate semi-diameters.*

STEP 4 (eliminate Pv and FP)

If we now use equation (10) to substitute for Pv in equation (9) and rearrange, we obtain,

$$\frac{QR}{QT^2} = \frac{CP}{CD^2} \cdot \frac{CA^3}{FP^2} \cdot \frac{Qv^2}{Gv \cdot Qx^2} \tag{11}$$

But, areas of parallelograms formed from conjugate diameters are equal [17, p166][15, p53],

$$\therefore CD \cdot FP = CA \cdot CB\,. \tag{12}$$

Thus,

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{Qv^2}{Gv \cdot Qx^2}. \tag{13}$$

STEP 5 (define QR/QT^2 in terms of properties of ellipse, when P and Q coincide)

It is seen from figure (5) that when P and Q are coincident,

$$Gv = 2\,CP \tag{14}$$

and,

$$Qv = Qx\,. \tag{15}$$

Therefore, under these conditions equation (13) reduces to,

$$\frac{QR}{QT^2} = \frac{CA}{2\,CB^2}. \tag{16}$$

This step is similar to that which Newton took in Proposition XI, Theorem VI. Newton did not use explicitly the notion of a limit in this calculation (or, *ultimate ratio* as he referred to this process), though it is likely that he thought in these terms when formulating his solution to the problem.

[This step will seem crude and non-rigorous to the modern reader and, furthermore, in many situations this approach could lead to erroneous results. But, in Newton's time it was a bold and farsighted step which opened up new vistas for both mathematics and physics. We can, of course, modernise step 5 to introduce the explicit use of a limit, as shown below. However, whilst Newton did present the limit process in somewhat unsophisticated terms, we have to bear in mind that this was written in 1686 and that our ideas of 'rigor', which embody relatively modern concepts, had not yet been formulated [13]].

Alternative STEP 5 (a more modern form, with the explicit use of a limit)

It is seen from figure (5) that,

$$Qv = Qx + xv \tag{17}$$

and,

$$Gv = 2\,CP - Pv \tag{18}$$

$$\therefore \frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{(Qx + xv)^2}{(2\,CP - Pv) \cdot Qx^2}. \tag{19}$$

It is at this point that the concept of a limit has to be introduced to enable the final crucial step to be taken, i.e. we let Q tend to P and thus,

$$Q \to P \Rightarrow \begin{cases} xv \to 0 \\ Pv \to 0 \end{cases} \tag{20}$$

and, in the limit when Q and P are coincident,

$$\frac{QR}{QT^2} = \frac{CA}{2\,CB^2}\;. \tag{21}$$

Newton took an equivalent step in Proposition XI, Theorem VI.

STEP 6 (show that QR/QT^2 is constant)

From the theory of conic sections [17], the R.H.S. of equation (16), i.e. $2\,CB^2/CA$, is identically equal to the reciprocal of the *principal latus rectum* of an ellipse,

$$\therefore \frac{QR}{QT^2} = \frac{1}{L} = constant, \tag{22}$$

where,

L = principle latus rectum of the ellipse, $[m]$.

STEP 7 (demonstrate Inverse Square Law)

For any orbit where the centripetal force is always directed towards a single point, the force is described by equation (5). Thus, it follows directly from the result of equation (22) that,

$$f \propto \frac{1}{SP^2}. \tag{23}$$

This is the desired result, i.e., *if a body moves in an elliptical orbit subject only to a gravitational force acting towards one focus, then this gravitational force must vary inversely as the square of the distance between the body and the focus.*

END

He also showed that all these results apply equally to any other type of conic-section orbit. Further, to show that he understood the problem in its entirety, and that he had solved it completely, he included the analysis for a wide range of problems involving various different laws of gravity.

In *Book III* of the *Principia* entitled *System of The World*, Newton illustrates his usual clear thinking by describing experiments he carried out in connection with the dynamic performance of a pendulum when filled with differing materials. He concluded that the motion was the same regardless of material ("*I tried experiments with gold, silver, lead, glass, sand, common salt, wood, water, and wheat*"), and was dependent solely upon the quantity of matter, which he measured meticulously: "*... I could manifestly have discovered a difference in matter less than one thousandth of the whole, had any such been*"[15, p411]. A direct consequence of this observation by Newton, is that it leads us immediately from equation (23) to the modern form of his *Universal Law of Gravitation*, i.e.

$$Force = -\frac{GmM}{r^2} \quad [N], \tag{24}$$

where,

G = universal gravitational constant, $[6.67259 \times 10^{-11} Nm^2/kg^2]$
m = mass of body in orbit, $[kg]$
M = mass of primary body, $[kg]$
r = distance between centres of m and M, $[m]$

The minus sign is there by convention, indicating an attractive force.

The derivation of equation (24) is described lucidly by I. B. Cohen in his book, *Birth Of a New Physics* [2].

6 Hyperbolic and Parabolic Orbits

Newton used almost identical methods in the Principia to demonstrate that hyperbolic and parabolic orbits also require an Inverse Square Law of gravity; and these methods were equally abstruse. A similar approach to that given above for elliptical orbits (see Parts II and III) also leads to more natural geometrical solutions to these problems than those included as Proposition XII, Problem VII and Proposition XIII, Problem VIII of the *Principia*[10].

7 Conclusion

A new and more straight forward derivation for the inverse square law of gravity has been presented in steps 1 to 7 above. It is believed that this derivation represents a realistic representation of what Newton's original thought processes could have been when he first solved this unique and interesting problem of physics. The assertion that the discovery process is most likely be different from the final printed form is justified because procedures of discovery are commonly different from procedures of proof which have to convince a professional audience[16].

Whilst we can only speculate as to why Newton made his derivation artificially abstruse, it is well known that he did not like controversy or being "*baited by little Smatterers in Mathematicks...*" [19, p459]. However, it is possible that because this particular problem was the most important contained within the *Principia*, he did expend some effort to make the solution difficult to follow. This, the argument goes, would have the effect of enabling only the most competent of mathematical philosophers to enter into a public dialogue as to the correctness of his work. As it happened, very few scientifically sophisticated people attempted to refute Newton's results.

Step 5 is the only part in the above derivation which does not conform to the application of traditional geometry as understood by Newton's contemporaries but, I submit, it does **NOT** constitute an application of the calculus. This supports the view of D. T. Whiteside[20] that Newton did arrive at these results by purely geometric considerations. Further, the geometric approach as presented above with the modified step 5, is very straight forward and offers a simplified introduction to the theory of orbits. It can be shown that parabolic and hyperbolic orbits also yield to a similar geometric analysis.

It is well known that average students of science and mathematics encounter some difficulty when they are first introduced to the theory of orbits. This introduction is likely to be less traumatic if they are first presented with the theory by means of a geometric approach similar to that given above, rather than one which relies strictly on the calculus. I feel that, not only would this enable the history of the subject to be presented in a very natural way but, the student is likely to gain a better insight into the problem and its solution; thus, being more ready to move on to apply the methods of the calculus.

In assessing the particular work of Newton discussed above, it is chastening to remember that it represents only a very small proportion of that contained within the *Principia*, let alone his other work in the fields of mathematics and the theory of light [19]. Newton was also a practical man who was able to invent and construct both a new sextant and the reflecting telescope.

[10] Newton also knew that, not only did the value of QR/QT2 have to be constant under an inverse square law of gravity but, the types of orbit possible are determined by its limit. He had already drafted text to this effect[14,pp555] for inclusion in a later revision of the Principia, "... then this body shall move in some conic whose latus rectum is the ultimate value of the quantity QT 2/QR when the elements PR, QR are infinitely diminished." However, this revision never came to print.

Part II - Hyperbolic Orbits

8 Newton's Derivation of the Inverse Square Law

The following derivation (in box) is as set out in Cajori's revised version of Motte's translation of the third edition of Newton's Principia [15, pp57-9]. Figure (6), has been drawn to correspond to the figure of Proposition XII, Problem VII included in reference [15, p58] (except for figure caption). However, in figure (6) RQ is shown as being parallel to SP, which is correct, whereas it is shown incorrectly in reference [15]. It should also be noted that, whilst SP appears to be at right angles to AC, this is not a necessary condition for the analysis.

PROPOSITION XII. PROBLEM VII

Suppose a body to move in a hyperbola; it is required to find the law of the centripetal force tending to the focus of that figure.

Let CA, CB be the semiaxis of the hyperbola; PG, KD other conjugate diameters; PF a perpendicular to the dameter KD; and Qv an ordinate to the diameter GP. Draw SP cutting the diameter DK in E, and the ordinate Qv in x, and complete the paralelogram $QRPx$. It is evident that EP is equal to the semitransverse axis AC; for drawing HI, from the other focus H of the hyperbola, parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half difference of PS, PI; that is (because of the parallels IH, PR, and the equal angles IPR, HPZ), of PS, PH, the difference of which is equal to the whole axis $2AC$. Draw QT perpendicular

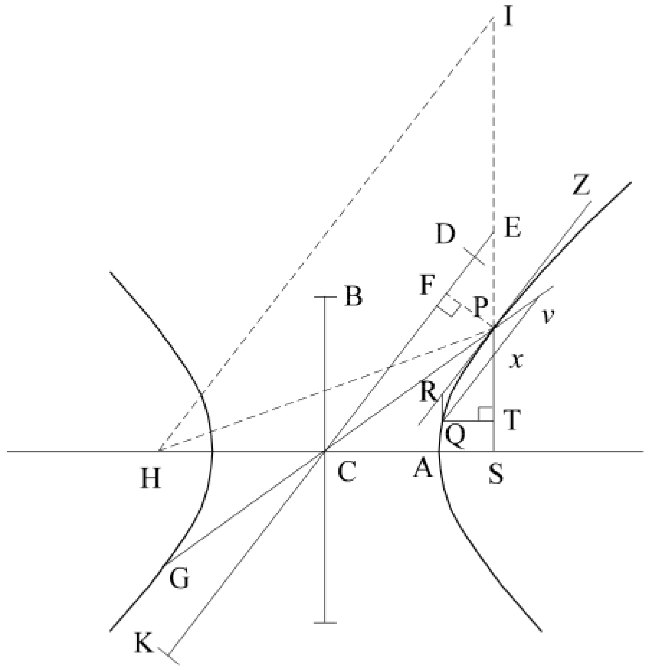

Figure 6: Geometry Of A Hyperbolic Orbit

to SP; and putting L for the principle latus rectum of the hyperbola (that is, for $\frac{2\,BC^2}{AC}$), we shall have

$$L\cdot QR : L\cdot Pv = QR : Pv = Px : Pv = PE : PC = AC : PC,$$

also, $L\cdot Pv : Gv\cdot Pv = L : Gv$, and $Gv\cdot Pv : Qv^2 = PC^2 : CD^2$. By Cor. II, Lem. VII, when P and Q coincide, $Qx^2 = Qv^2$, and Qx^2 or $Qv^2 : QT^2 = EP^2 : PF^2 = CA^2 : PF^2$, by Lem. XII, $= CD^2 : CB^2$. Multiplying together corresponding terms of the four proportions, and simplifying,

$$L\cdot QR : QT^2 = AC\cdot L\cdot PC^2\cdot CD^2 : PC\cdot Gv\cdot CD^2\cdot CB^2 = 2\,PC : Gv,$$

since $AC\cdot L = 2\,BC^2$. But the points Q and P coinciding, $2\,PC$ and Gv are equal. And therefore the quantities $L\cdot QR$ and QT^2, proportional to them, will also be equal. Let those equals be drawn into $\frac{SP^2}{QR}$, and we shall have $L\cdot SP^2$ equal to $\frac{SP^2\cdot QT^2}{QR}$. And therefore (by Cor. I and V, Prop. VI) the centripetal force is inversely as $L\cdot SP^2$, that is inversely as the square of the distance SP. Q.E.I.

9 An Alternative Derivation of the Inverse Square Law

We are now in a position to look at an alternative to Newton's original derivation of the inverse square law based upon a hyperbolic orbit.

Consider a body P moving in a hyperbolic orbit about a fixed point S, as shown in figure (6). The body is assumed to be moving along the hyperbolic curve subject only to a gravitational force directed towards the focus, S. We note that the hyperbola has semi-diameters AC and BC, and conjugate diameters KD and GP; the line RZ forms a tangent to the hyperbola at point P, and HI and Qv are parallel to RZ; IP is an extension to SP, DE is an extension to KD meeting IP at E and Pv is an extension to GP meeting Qv at v; Qv and SP cross at point x; QT is normal to SP and FP is normal to RZ.

The derivation given below (using modern mathematical notation) forms a natural calculation sequence, and, I believe, is more likely to be the route by which Newton first obtained his solution rather than either, that presented in the *Principia* or, one he would have found by using the calculus.

For any orbit where a body is subject only to a force of gravity directed towards a fixed point S, it has been shown in Part I that, regardless of the particular form of gravitational law,

$$f \propto \frac{QR}{SP^2 \cdot QT^2} \tag{25}$$

where the force f has been defined previously.

Thus, in order to show that the gravitational force for a hyperbolic orbit conforms to an inverse square law, it is only necessary to show that the ratio of QR to QT^2 remains constant regardless of the position of P. This is demonstrated below.

STEP 1

It is seen from figure (6) that $\triangle xPv$ is similar to $\triangle CPE$ and that its shape is independent of the position of Q.

$$\therefore Px = QR = Pv \cdot \frac{PE}{PC}. \tag{26}$$

But, due to similar triangles IE is equal to ES,

$$IP - PS = 2\,EP \tag{27}$$

For a hyperbola, angles $\widehat{HPZ}$ and $\widehat{IPR}$ are equal because the tangent RZ is the internal bisector of angle $\widehat{HPS}$. Thus, as HI is parallel to CE,

$$HP = IP \tag{28}$$

$$HP - PS = 2\,EP = 2\,AC = 2a \tag{29}$$

Therefore, it follows by definition that,

$$QR = Pv \cdot \frac{AC}{PC}. \tag{30}$$

STEP 2

It is seen from figure (6) that $\triangle QTx$ is similar to $\triangle EFP$ and that its shape is independent of the position of Q.

$$QT = Qx \cdot \frac{FP}{EP} = Qx \cdot \frac{FP}{AC} \, . \tag{31}$$

STEP 3

From equations (30) and (31) it is seen that,

$$\frac{QR}{QT^2} = Pv \cdot \frac{AC}{PC} \cdot \frac{AC^2}{Qx^2 \cdot FP^2} \tag{32}$$

$$\therefore \frac{QR}{QT^2} = \frac{Pv}{Qx^2} \cdot \frac{AC^3}{PC \cdot FP^2} \tag{33}$$

STEP 4

From the Appendix it is seen that,

$$\frac{Gv \cdot Pv}{Qv^2} = \frac{CP^2}{CD^2}. \tag{34}$$

If we now use equation (34) to substitute for Pv into equation (32) and rearrange, we obtain,

$$\frac{QR}{QT^2} = \frac{CP}{CD^2} \cdot \frac{CA^3}{FP^2} \cdot \frac{Qv^2}{Gv \cdot Qx^2} \tag{35}$$

But, areas of parallelograms formed from conjugate diameters are equal [17, p194][15, p53],

$$\therefore CD \cdot FP = CA \cdot CB \, . \tag{36}$$

Thus,

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{Qv^2}{Gv \cdot Qx^2}. \tag{37}$$

STEP 5

It is seen from figure (6) that,

$$Qv = Qx + xv \tag{38}$$

$$\therefore \frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{(Qx + xv)^2}{(2\,CP - Pv) \cdot Qx^2}. \tag{39}$$

If we let Q tend to P then $Qv \to Qx$, or using the same approach as in Part I,

$$Q \to P \Rightarrow \begin{cases} xv \to 0 \\ Pv \to 0 \end{cases} \tag{40}$$

and, in the limit when Q and P are coincident,

$$\frac{QR}{QT^2} = \frac{CA}{2\,CB^2} \, . \tag{41}$$

From the theory of conic sections [17, p187], the R.H.S. of equation (41), i.e. $2\,CB^2/CA$, is identically equal to the reciprocal of the *latus rectum* of the hyperbola,

$$\therefore \frac{QR}{QT^2} = \frac{1}{L} = constant, \tag{42}$$

where,

L= latus rectum of the hyperbola.

STEP 6

For any orbit where the centripetal force is always directed towards a single point, the force is described by equation (25). Thus, it follows directly from equation (42) that,

$$f \propto \frac{1}{SP^2}. \tag{43}$$

Which is the desired result, i.e., if a body moves in a hyperbolic orbit subject only to a gravitational force acting towards one focus, then this gravitational force must vary inversely as the square of the distance between the body and the focus. This is the result that Newton arrived at in Proposition XII, Problem VII of the Principia [15, p58].

Part III - Parabolic Orbits

10 Newton's Derivation of the Inverse Square Law

The following derivation (in box) is as set out in Cajori's revised version of Motte's translation of the third edition of Newton's Principia [15, pp60-1]. Figure (7), has been drawn to correspond to the figure of Proposition XIII, Problem VIII included in reference [15, p60] (except for figure caption).

PROPOSITION XIII. PROBLEM VIII

If a body moves in the perimeter of a parabola; it is required to find the law of the centripetal force tending to the focus of that figure.

Retaining the construction of the preceding Lemma, let P be the body in the perimeter of the parabola; and from the place Q, into which it is next

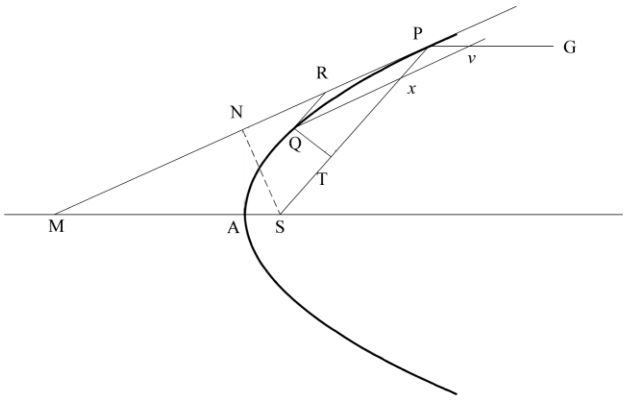

Figure 7: Geometry of a parabolic orbit

to succeed, draw QR parallel and QT perpendicular to SP, as also Qv parallel to the tangent, and meeting the diameter PG in v, and the distance SP in x. Now, because of the similar triangles Pvx, SPM, and of the equal sides SP, SM of the one, the sides Px or QR and Pv of the other will be also equal. But (by the conic sections) the square of the ordinate Qv is equal to the rectangle under the latus rectum and the segment Pv of the diameter; that is (by Lem. XIII), to the rectangle $4PS \cdot QR$; and the points P and Q coinciding, (by Cor. II, Lem. VII), $Qx = Qv$. And therefore Qx^2, in this case, becomes equal to the rectangle $4PS \cdot QR$. But (because of the similar triangles QxT, SPN),

$$Qx^2 : QT^2 = PS^2 : SN^2 = PS : SA, \text{ (by Cor. I, Lem. XIV)},$$

$$= 4PS \cdot QR : 4SA \cdot QR.$$

Therefore (by Prop. IX, Book V, *Elem. of Euclid*), $QT^2 = 4SA \cdot QR$. Multiply these equals by $\dfrac{SP^2}{QR}$, and $\dfrac{SP^2 \cdot QT^2}{QR}$ will become equal to $SP^2 \cdot 4SA$: and therefore (by Cor. I and V, Prop. VI), the centripetal force is inversely as $SP^2 \cdot 4SA$; that is, because $4SA$ is given, inversely as the square of the distance SP. Q.E.I.

11 An Alternative Derivation of the Inverse Square Law

Consider a body P, moving in a parabolic orbit about a fixed point S, as shown in figure (7).

The body is assumed to be moving along the parabolic curve subject only to a gravitational force directed towards the focus, S. The line MP forms a tangent to the parabola at point P and meets the x-axis MA, at point M. A line NA will also form a tangent to the parabola, but at the vertex, A. The line PG is parallel to the x-axis and the line Qv is parallel to MP.

For any orbit where a body is subject only to a force of gravity directed towards a fixed point S, it has been shown in Part I that, regardless of the particular form of gravitational law,

$$f \propto \frac{QR}{SP^2 \cdot QT^2} \tag{44}$$

where force f has been defined previously.

Thus, in order to show that the gravitational force for a parabolic orbit conforms to an inverse square law, it is only necessary to show that the ratio of QR to QT^2 remains constant irrespective of the position of Q. This is demonstrated below.

STEP 1

It is seen from figure (7) that $\triangle xPv$ is similar to $\triangle MSP$ and that its shape is independent of the position of Q.

$$\therefore Px = QR = Pv \cdot \frac{SP}{SM}. \tag{45}$$

but, for a parabola, SP is equal to SM,

$$\therefore QR = Pv. \tag{46}$$

STEP 2

$$\therefore QT = Qx \cdot \frac{SN}{SP} \tag{47}$$

$$\therefore \frac{QR}{QT^2} = \frac{Pv}{\left(Qx \cdot \dfrac{SN}{SP}\right)^2} = \frac{Pv}{Qx^2} \cdot \frac{SP^2}{SN^2} \tag{48}$$

It is seen from figure (7) that $\triangle QTx$ is similar to $\triangle SNP$ and that its shape is also independent of the position of Q.

STEP 3

Again, from figure (7) it is seen that $\triangle ANS$ is similar to $\triangle SNP$ and that its shape is independent of the position of Q.

$$\therefore \frac{SP}{SN} = \frac{SN}{SA} \tag{49}$$

$$\therefore \frac{SP^2}{SN^2} = \frac{SP}{SA}. \tag{50}$$

Thus, from equations (48) and (50) we obtain,

$$\frac{QR}{QT^2} = \frac{Pv}{Qx^2} \cdot \frac{SP}{SA}. \tag{51}$$

STEP 4

The standard form of equation for a parabola is

$$y = 4ax^2, \tag{52}$$

where a represents the distance from the origin to the focus, and x and y are co-ordinates on the axes.

Now, from the fundamental geometry of conic-sections [17, p122], it is known that a parabola can be modified by a simple linear transformation in x and y to obtain

$$Y^2 = 4a'X, \tag{53}$$

where, a' is the distance from the transformed origin to the focus and X and Y represent co-ordinates on the transformed axes.

In figure (7) we can consider the transformed parabola to be represented by,

P = new origin
MP = new Y axis
PG = new X axis
SP = a', distance from new origin to the focus (fixed)

Thus, the point Q on the new axis is represented by,

$$Qv^2 = 4SP \cdot Pv\,. \tag{54}$$

Therefore, on substituting equation (54) into (51) we obtain,

$$\frac{QR}{QT^2} = \frac{Qv^2}{4SA \cdot Qx^2}. \tag{55}$$

STEP 5

It is seen from figure (7) that,

$$Qv = Qx + xv \tag{56}$$

$$\therefore \frac{QR}{QT^2} = \frac{(Qx + xv)^2}{4SA \cdot Qx^2}. \tag{57}$$

If we let Q tend to P, then $Qv \to Qx$, or using the same approach as in Part I,

$$Q \to P \quad \Rightarrow \quad xv \to 0 \tag{58}$$

and, in the limit when Q and P are coincident,

$$\frac{QR}{QT^2} = \frac{Qx^2}{4SA \cdot Qx^2} = \frac{1}{4SA}. \tag{59}$$

But $4SA$ is identically equal to the *latus rectum* of a parabola,

$$\therefore \frac{QR}{QT^2} = \frac{1}{L} = constant. \tag{60}$$

where,

L = latus rectum of the parabola.

STEP 6

For any orbit where the centripetal force is always directed towards a single point, the force is described by equation (44). Thus, it follows directly from equation (60) that,

$$f \propto \frac{1}{SP^2} \tag{61}$$

Which is the desired result, i.e., if a body moves in a parabolic orbit subject only to a gravitational force acting towards one focus, then this gravitational force must vary inversely as the square of the distance between the body and the focus. This is the result that Newton arrived at in Proposition XIII, Problem VIII of the Principia [15, pp60-1].

Appendix I - Kepler's Three Laws Of Planetary Motion

Kepler's first and second laws of motion were set down for the first time in the *Astronomia nova* in 1609:

FIRST LAW: *The Orbits of The Planets are ellipses with The Sun in a common focus;*

SECOND LAW: *The line joining a Planet to the Sun sweeps out equal areas in equal times.*

Kepler's third law was set down for the first time in the *Harmonice mundi* in 1619:

THIRD LAW: *The squares of the periodic times are proportional to the cubes of the mean distances from The Sun.*

Appendix II - Newton's Three Laws Of Motion

Newton's three laws of motion were set down for the first time in *The Principia* in 1687:

THE FIRST LAW: *Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it;*

THE SECOND LAW: *The change in motion is proportional to the motive force impressed; and is made in the direction of the right line in which the force is impressed;*

THE THIRD LAW: *To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.*

Appendix III - Conjugate Diameter Equation Of An Ellipse

An ellipse can be described mathematically by the following fundamental equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{62}$$

where a and b represent the semi-major and semi-minor axes respectively. Equation (62) can be modified by a simple linear transformation in x and y to obtain,

$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1, \tag{63}$$

where a' and b' represent the conjugate semi-diameters, i.e. CP and CD in figure (5), and X and Y represent co-ordinates on the transformed axes GP and DK.

Thus, it is seen from figure (5) and equation (20) that,

$$\frac{Gv \cdot Pv}{Qv^2} = \frac{(a' + X)\,(a' - X)}{Y^2} = \frac{a'^2 - X^2}{Y^2} = \frac{a'^2}{b'^2} = \frac{CP^2}{CD^2} \tag{64}$$

This result was readily known to Newton [19, p56] and his contemporaries, and can be found in most text books dealing with elementary co-ordinate geometry [17, pp170-1].

Appendix IV - Conjugate Diameter Equation Of A Hyperbola

A hyperbola can be described mathematically by the following fundamental equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{65}$$

where a and b represent the semi-major and semi-minor axes respectively. Equation (65) can be modified by a simple linear transformation in x and y to obtain,

$$\frac{X^2}{a'^2} - \frac{Y^2}{b'^2} = 1, \tag{66}$$

where a' and b' represent the conjugate semi-diameters, i.e. CP and CD in figure (6), and X and Y represent co-ordinates on the transformed axes GP and KD.

Thus, it is seen from figure (6) and equation (40) that,

$$\frac{Gv \cdot Pv}{Qv^2} = \frac{(X + a')(X - a')}{Y^2} = \frac{X^2 - a'^2}{Y^2} = \frac{a'^2}{b'^2} = \frac{CP^2}{CD^2} \tag{67}$$

This result was readily known to Newton [15, p58] and his contemporaries, and can be found in most text books dealing with elementary co-ordinate geometry [17, p196].

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