Preface

Partial differential equations (PDEs) have been developed and used in science and engineering for more than 200 years, yet they remain a very active area of research because of both their role in mathematics and their application to virtually all areas of science and engineering. This research has been spurred by the relatively recent development of computer solution methods for PDEs. These have extended PDE applications such that we can now quantify broad areas of physical, chemical and biological phenomena. The current development of PDE solution methods is an active area of research that has benefited greatly from advances in computer hardware and software, and the growing interest in addressing PDE models of increasing complexity.

A large class of models now being actively studied are of a type and complexity such that their solutions are usually beyond traditional mathematical analysis. Consequently, numerical methods have to be employed. These numerical methods, some of which are still being developed, require testing and validation. This is often achieved by studying PDEs that have known exact analytical solutions. The development of analytical solutions is also an active area of research, with many advances being reported recently, particularly for systems described by nonlinear PDEs. Thus, the development of analytical solutions directly supports the development of numerical methods by providing a spectrum of test problems that can be used to evaluate numerical methods.

This book surveys some of these new developments in analytical and numerical methods and is aimed at senior undergraduates, postgraduates and professionals in the fields of engineering, mathematics and the sciences. It relates these new developments through the exposition of a series of solutions to complex PDE problems. The PDEs that have been selected are largely *named* in the sense that they are generally closely linked to their original contributors. These names usually reflect the fact that the PDEs are widely recognized and are of fundamental importance to the understanding of many application areas. Each chapter follows the general format:

- The PDE and its associated auxiliary conditions (initial conditions (ICs) and boundary conditions (BCs)) are stated.

- A series of routines is discussed with detailed explanations of the code and how it relates to the PDE. They are written in Matlab, but have been specifically programmed so that they can be easily converted to equivalent routines in other languages. The routines have the following common features:

  – The numerical procedure is the method of lines (MOL) in which the boundary value (spatial) partial derivatives are replaced with algebraic approximations, in the present case finite differences (FDs), although other approximations could be used such as finite elements (FEs), finite volumes (FVs), and spectral methods (SMs). The FD approximations are implemented in a series of library routines; the details of how these routines were developed are given as an introduction to facilitate the development of new routines that may be required for particular PDE applications.

  – The resulting system of ordinary differential equations (ODEs) in an initial value variable, typically time in an application, is then integrated numerically using an initial value ODE integrator from the Matlab library.

  – The displayed numerical output also includes the analytical solution and the difference between the numerical and analytical solutions. The agreement between the two solutions is displayed numerically and graphically as a way of demonstrating the validity of the numerical methods.

- An analytical solution for the PDE is stated, including a reference to the original source of the solution, and in some cases, a verification (proof) of the solution by substitution into the PDE and auxiliary conditions.

- Additionally, in several chapters, the analytical solution is derived by relatively new techniques such as the *tanh*, *exp*, *Riccati* or *factorization* based methods. The derivation is either by direct application of the analytical method or through the use of the *computer algebra system* (CAS), Maple. Where Maple is used, the associated code is included in the text along with a description of its main functional elements. This code usually demonstrates the use of our new Maple procedures which implement various analytical methods that are described in the text. Graphical output from these Maple applications is provided, including a 2D animation (to facilitate insight into and understanding of the solution) and a plot in 3D perspective. Maple is also used in other chapters to confirm analytical solutions from the literature. Where appropriate, the code is provided in the *mws* file format as well as the *mw* format so that it will also run in early versions of Maple.

- The form of the analytical solution is considered, with particular emphasis on traveling wave analysis by which the PDE (in an Eulerian or fixed frame) is converted to an ODE (in a Lagrangian or moving frame). An analytical solution to the ODE is then derived and the inverse coordinate transformation is applied to provide an analytical solution to the PDE.

- A second approach to a PDE analytical solution, the method of residual functions, is also used in some of the chapters to derive an analytical solution to a PDE that is closely related to the original PDE.

- The basic approach of traveling wave analysis, whereby a PDE is transformed to an associated ODE, is also reversed in two chapters. These start with ODEs that are then restated as PDEs that are first and second order in the initial value variable. The analytical solution to the initial ODE is then provided as an analytical solution to the PDE.

- The structure of the PDE is usually revisited briefly with regard to its form, such as whether it is first or second order in the initial value variable, the order of the boundary value derivatives, the features of nonlinear terms and the form of the BCs. In this way, the intention of the final summary is to suggest concepts and computational approaches that can be applied in new PDE applications.

- Each chapter concludes with a discussion of the numerical solution, particularly how it conforms to the initial statement of the PDE and its auxiliary conditions; also, the numerical solution is evaluated with regard to the magnitude of the errors and how these errors might be reduced through additional computation.

- In Chapter 2 we discuss the linear advection equation, one of the simplest PDEs, and show that solutions involving steep gradients or discontinuities can be difficult to achieve numerically. We then illustrate how flux limiters can be employed to improve the fidelity of the numerical solution. A short appendix to this chapter is also included which briefly discusses some of the background to the ideas behind flux limiters.

- A general appendix details the *tanh*, *exp*, *Ricatti*, *direct integration* and *factorization* based methods. Maple implementation, by way of newly developed procedures, is included for the tanh, exp and Ricatti based analytical methods. As referred to above, these general features are then referenced for specific applications in appendices to individual chapters.

In summary, the major focus of this book is the numerical MOL solution of PDEs and the testing of numerical methods with analytical solutions, through a series of applications. The origin of the analytical solutions through traveling wave and residual function analysis provides a framework for the development of analytical solutions to nonlinear PDEs that are now widely reported in the literature. Also, in selected chapters, procedures based on the tanh, exp and Ricatti methods that have recently received major attention are used to illustrate the derivation of analytical solutions. References are provided where appropriate to additional information on the techniques and methods deployed.

Our intention is to provide a set of software tools that implement numerical and analytical methods that can be applied to a broad spectrum of problems in PDEs. They are based on the concept of a traveling wave and the central feature of these methods is conversion of the system PDEs to ODEs. The discussion is limited to one dimensional (1D) PDEs and complements our earlier book: *A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab*, Cambridge University Press, 2009.

Finally, all the code discussed in this book, along with a set of the MOL DSS library routines, is available for download from `www.pdecomp.net`

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June 1, 2010